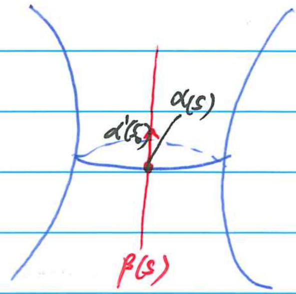
Assignment 6.

1.(a) Suppose at some S_0,



$$u'(S_0)=0$$

$$\therefore \alpha'(S_0) = V'(S_0)\cdot X_v$$

Let $\beta(s)$ be the meridian with $u=u(S_0)$

and $\beta(s)$ is parametrized by arc-length.

$$\therefore \quad \beta(S_0)=\alpha(S_0), \quad \beta'(S_0)=\alpha'(S_0)$$

By the uniqueness of geodesic

$$\beta(S)=\alpha(S)$$

which means $\alpha(S)$ is a meridian. But this is a contradiction.

$\therefore u'(s)\neq 0$ for any s.

$\therefore$ $S=S(u)$ can be expressed as a function of u locally.

(b) $\because \alpha$ is a geodesic on a revolution surface

$\therefore$ $r(s)\cdot \sin\theta(s)$ is a constant

$$= f\cdot \langle \alpha', \frac{X_u}{|X_u|}\rangle$$

$$= f\cdot u'|X_u|$$

$$= f^2 u' = c \text{ for some constant } c.$$

and $f = r \geq |r\sin\theta| = |c|$

$$\therefore u' = \frac{c}{f^2}$$

$\because \alpha$ is parametrized by arc-length

$$\therefore \quad |\alpha'|^2 = 1$$

$$|u'X_u + v'X_v|^2 = 1$$

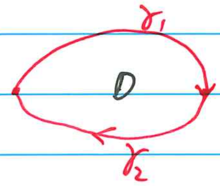
$$f^2(u')^2 + (f_v^2+g_v^2)(v')^2 = 1$$

$$\therefore (v')^2 = \frac{1-f^2(u')^2}{f_v^2+g_v^2} = \frac{1-\frac{c^2}{f^2}}{f_v^2+g_v^2} = \frac{f^2-c^2}{f^2(f_v^2+g_v^2)}$$

$$\therefore \frac{dv}{du} = \frac{v'}{u'} = \pm\frac{1}{f}\sqrt{\frac{f^2-c^2}{f_v^2+g_v^2}}\cdot\frac{f^2}{c} = \pm\frac{f}{c}\sqrt{\frac{f^2-c^2}{f_v^2+g_v^2}}.$$

3、 Suppose NOT.

Then we will have two geodesics and they bound an open set which is diffeomorphic to a disk.



$\therefore$ By the Gauss-Bonnet Theorem,

$$\sum_{i=1}^{2} \int_{\gamma_i} k_g ds + \iint_D K dA + \sum_{j=1}^{2} \theta_j = 2\pi \cdot \chi(D)$$

$$0 + \iint_D K dA + \sum_{j=1}^{2} \theta_j = 2\pi \cdot 1$$

$\therefore 2\pi \leq 0 + 0 + 2\pi$ (By assumption, $K \leq 0$)

$\therefore K \equiv 0$ and $\theta_1 = \theta_2 = \pi$

$\therefore$ By the uniqueness of geodesic, $\gamma_1 = -\gamma_2$, which is a contradiction to γ_1 and γ_2 bound an open set.

4. (a) $\because$ $$\iint_M K dA = 2\pi \cdot \chi(M) = 4\pi(1-g)$$

if M is not homeomorphic to a sphere then $g \geq 1$

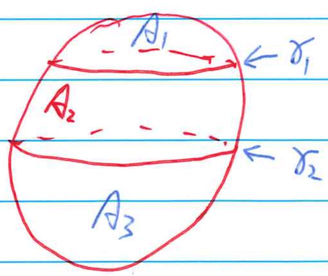
$\therefore 0 \geq 4\pi(1-g) = \iint_M K dA > 0$ since by assumption $K > 0$

which is a contradiction.

$\therefore M$ is homeomorphic to a sphere.

(b) Suppose we have two distinct simple closed geodesics on M, and they do NOT intersect.

$\therefore \gamma_1, \gamma_2$ devide M into three parts.



$\therefore$ $$\int_{\gamma_1} \cancelto{0}{k_g} + \int_{A_1} K dA = 2\pi \cdot \chi(A_1) = 2\pi$$

$$\int_{\gamma_2} \cancelto{0}{k_g} + \int_{A_3} K dA = 2\pi \cdot \chi(A_3) = 2\pi$$

$\therefore$ $$\int_{A_1} K + \int_{A_3} K = 4\pi$$

$\because$ $$\int_M K = 2\pi \cdot \chi(M) = 4\pi$$

$\therefore$ $$\int_{A_3} K = 4\pi - 4\pi = 0$$

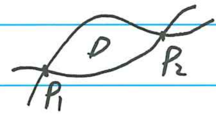
But $K > 0$, this is a contradiction.

$\therefore$ γ_1 and γ_2 must intersect.

5. Suppose M has two closed geodesic γ_1, γ_2

Case 1: If γ_1 and γ_2 have more than one intersection.

Then they form a region D which is homeomorphic to a disk.



But by Q3, this is impossible.

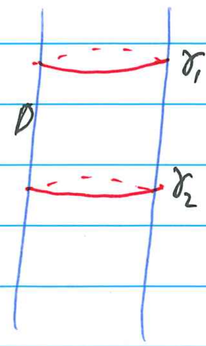
Case 2: If γ_1 and γ_2 have only one intersection.

Then they must be tangent to each other at the common point.

So by the uniqueness of geodesic, $\gamma_1 = \gamma_2$.

Case 3: If γ_1 and γ_2 have no intersection.

and they bound a region D



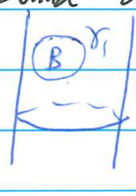
$\therefore$ By the Gauss-Bonnet Theorem

$$\sum_{i=1}^{2} \int_{\gamma_i} \cancelto{0}{k_g} + \int_D K = 2\pi\chi(D) = 0$$

But by assumption $K < 0$, so it is a contradiction.

Case 4: If γ_1 and γ_2 have no intersection and they don't bound any region.

then one of them bound a region which is homeomorphic to a disk.



By Gauss-Bonnet Theorem

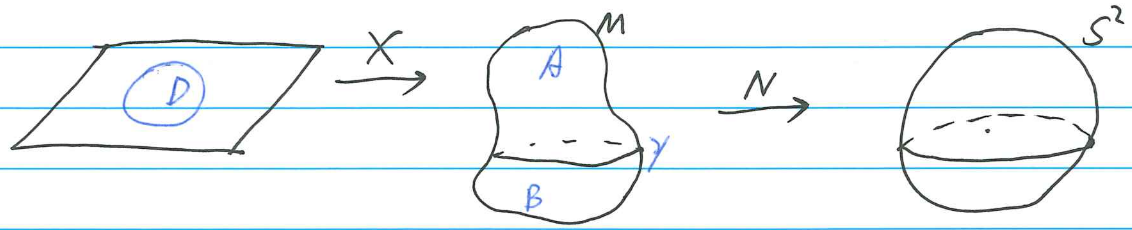
$$\int_{\gamma_1} \cancel{k_g}^{\,0} + \int_B K = 2\pi \chi(B) = 2\pi$$

But by assumption, $K<0$.

So it is a contradiction.

$\therefore$ M has at most one closed geodesic.

6.



let X be a parametrization of M.

let N be the Gauss map. Then $N \circ X(u,v) \in S^2$ the unit sphere

$\therefore$ Area $(N \circ X(D))$

$$= \int_D |(N\circ X)_u \times (N \circ X)_v| \, du dv$$

$$= \int_D |dN(X_u) \times dN(X_v)| du dv$$

$$= \int_D |S_p(X_u) \times S_p(X_v)| du dv \qquad S_p = \begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix}$$

$$= \int_D |(a_1^1 X_u + a_1^2 X_v) \times (a_2^1 X_u + a_2^2 X_v)| du dv$$

$$= \int_D |\det(S_p) X_u \times X_v| \, du dv$$

$$= \int_D |K| \cdot |X_u \times X_v| du dv$$

$$= \int_{X(D)} |K| \quad \text{for any } D \subseteq \mathbb{R}^2$$

$\therefore$ Area$(N(A))$

$= \int_A |K|$

$= \int_A K$ since $K > 0$

$= 2\pi \chi(A) - \int_\gamma \cancel{k_g}^{0}$

$= 2\pi.$

Area$(N(B))$

$= \int_B |K|$

$= \int_B K$

$= 2\pi \chi(B) - \int_\gamma \cancel{k_g}^{0}$

$= 2\pi$

$\therefore$ Area$(N(A))$ = Area$(N(B))$.